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Analytic Studies of the Long Range Beam-Beam Tune Shifts and Chromaticities

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Abstract

A formula is derived, which allows efficient analytical evaluation of the long range beam-beam tune shifts and chromaticities with amplitude. It assumes that the beams are infinitely short, oppositely charged, and with Gaussian transversal profile. The formula employs an infinite sum with favorable convergence rates, making it well suited especially for the long range case. Applications to the Tevatron are presented, including some proposed compensation schemes and their effect on the dynamic aperture.

1 INTRODUCTION

Beam-beam interactions play a major role in circular colliders, as, for example, the Tevatron's Run II [1]. Tune shifts with amplitude are used to quantitatively characterize the strength of these interactions, which can be head on, or long range. The amplitude (and parameter) dependent tune shifts to any order can be easily determined analytically from the map of a system using Differential Algebraic methods [2, 3], if the potential has a good polynomial approximation (usually the Taylor expansion). Unfortunately, the beam-beam potential does not admit a rapidly converging polynomial expansion for amplitudes of practical interest. As a consequence of this form of the potential, computation of the amplitude dependent tune shift often requires tracking and subsequent postprocessing. Therefore, an analytical formula would be useful for the fast evaluation of the tune shifts, and would provide insight into the structure of the beam-beam effects. Moreover, an analytic formula for the computation of the amplitude dependent tune shifts can be readily modified to provide a useful tool for the determination of the amplitude dependent chromaticities.

The expression for head on tune shift is well known [4], and an approximation for the long range tune shifts of round beams, which is valid in the large separation and small amplitude case, has been derived in [5]. In this note, we show that a formula can be derived for the tune shift that is always valid, and its evaluation is reduced to setting the truncation order in a reasonably fast converging infinite series, and a quadrature. All operations can be readily and quickly performed in, for example, Mathematica. The next few sections present the formulae, and the theory applied to the Tevatron. Some compensation schemes are proposed, and their effect on the dynamics is checked by tracking.

2 TUNE SHIFT AND CHROMATICITY FORMULAE

Technically, the amplitude dependent tune shift is the advance in angle along a torus in normal form space, where a particle moves with amplitude dependent frequency. Thus, the first step of the computation must be the transformation to normal form. In the Differential Algebraic picture, the transfer map is subjected to this transformation, while here, since the map is not easily computed, the transformation is applied directly to the Hamiltonian. Assuming a linearly dominated regime, it should be a very good approximation to make only a first order normal form transformation, and then take an average over the angles.

Assuming that the beam-beam interaction is the only perturbation to an otherwise simple harmonic motion with frequencies (ν_{x0}, ν_{y0}), the Hamiltonian is

$$H = \nu_{x0} J_x + \nu_{y0} J_y + U\left(J_x, \phi_x; J_y, \phi_y\right)\delta\left(\theta - \theta_c\right), \quad (1)$$

where $\delta(\theta)$ is the Dirac delta function, and θ is the independent variable. The delta function signifies that we neglect bunch length effects, and the interaction happens at a single collision point θ_c. Introducing the tune shift as $\Delta\nu_i = \nu_i - \nu_{i0}$, where i stands for x or y, from Hamilton's equations of motion we obtain that the average change in phase advance is given by the following formula:

$$2\pi\Delta\nu_i = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\partial U\left(J_x, \phi_x; J_y, \phi_y\right)}{\partial J_i} d\phi_x d\phi_y. \quad (2)$$

2.1 The Tune Shift Formula

For details on performing the integrals over the angles see [7]. The final expression for the horizontal amplitude dependent tune shift is

$$\Delta\nu_x = \frac{4\pi C}{\varepsilon_x}\int_0^1 \frac{e^{-(p_x+p_y)}}{v\left[v\left(r^2-1\right)+1\right]^{1/2}}\sum\nolimits_x\sum\nolimits_y dv, \quad (3)$$

where

$$\sum\nolimits_x = \sum_{k=0}^{\infty}\frac{\left(\frac{a_x}{d_x}\right)^k}{k!}\Gamma\left(k+\frac{1}{2}\right) \times \left[I_k\left(s_x\right)\left(\frac{2k}{a_x^2}-v\right)+I_{k+1}\left(s_x\right)\frac{s_x}{a_x^2}\right], \quad (4)$$

$$\sum\nolimits_y \quad = \quad \sum_{l=0}^{\infty} \frac{\left(\frac{a_y}{d_y}\right)^l}{l!} \Gamma\left(l+\frac{1}{2}\right) I_l\left(s_y\right). \qquad (5)$$

As shorthand notations we introduced the ratio of rms beam sizes $r = \sigma_y/\sigma_x$, and dimensionless variables for the amplitudes and separations according to $a_x = \sqrt{2\beta_x J_x}/\sigma_x$, $d_x = D_x/\sigma_x$ and similarly defined a_y and d_y. Using these notations, the following relationships have been used in (3): $p_x = v\left(a_x^2 + d_x^2\right)/2$, $r_x = v a_x^2/2$, $s_x = v a_x d_x$, $p_y = f v\left(a_y^2 + d_y^2\right)/2$, $r_y = f v a_y^2/2$, $s_y = f v a_y d_y$,where $f = \frac{r^2}{v(r^2-1)+1}$. The vertical amplitude dependent tune shift is derived analogously, due to symmetry in x and y.

In the zero amplitude ($a_x = a_y = 0$) and round beam ($r = 1$) case the integral can be done analytically to give (with $d^2 = d_x^2 + d_y^2$ and the beam-beam parameter ξ)

$$\lim_{a_x,a_y\to 0} \Delta\nu_x = \xi \frac{2}{d^4} \left\{ \begin{array}{c} e^{-\frac{d^2}{2}}\left[d_x^2 d^2 + \left(d_x^2 - d_y^2\right)\right] \\ -\left(d_x^2 - d_y^2\right) \end{array} \right\}. \qquad (6)$$

2.2 The Chromaticity Formula

It is rather straightforward to include chromatic effects into (3), to provide a formula for the computation of the chromaticities. To this end, we split the separation into two parts: one due to the closed orbits of on-momentum particles, the other due to dispersion for off-momentum particles. Denoting the dispersion (in units of rms beam size) at the location of the interaction by η, first we make the following replacements in (3): $d_x \mapsto d_x + \eta_x\delta$, $d_y \mapsto d_y + \eta_y\delta$, where δ is the relative momentum or energy deviation. By definition, the linear chromaticities are given by $Q_x^{'} = \frac{\partial\Delta\nu_x}{\partial\delta}\Big|_{\delta=0}$, $\quad Q_y^{'} = \frac{\partial\Delta\nu_y}{\partial\delta}\Big|_{\delta=0}$.

Using the symbolic capabilities of Mathematica, the derivative can be calculated symbolically, and then evaluated numerically. To this end, the horizontal chromaticity is given by

$$\begin{aligned} Q_x^{'} &= -\frac{2\pi C}{a_x\varepsilon_x}\int_0^1 \frac{e^{-(p_x+p_y)}}{\left[v\left(r^2-1\right)+1\right]^{1/2}} \\ &\times \sum_{k,l=0}^{\infty}\left[\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{k!l!}\left(\frac{a_x}{d_x}\right)^k\left(\frac{a_y}{d_y}\right)^l\right] \\ &\times \left[A I_l\left(s_y\right) + B I_{l+1}\left(s_y\right)\right] dv, \qquad (7) \end{aligned}$$

where

$$\begin{aligned} A &= 2d_x\left(d_x\eta_x + d_y\eta_y f\right) v I_{k-1}\left(s_x\right) \\ &- a_x\left(3d_x\eta_x + 2d_y\eta_y f\right) v I_k\left(s_x\right) \\ &+ 2\eta_x\left(a_x^2 v - k - 1\right) I_{k+1}\left(s_x\right) - s_x\eta_x I_{k+2}\left(s_x\right), \\ B &= 2a_y\eta_y f v\left(a_x I_k\left(s_x\right) - d_x I_{k-1}\left(s_x\right)\right). \qquad (8) \end{aligned}$$

The vertical chromaticity can be calculated similarly.

The chromaticity formula is greatly simplified in the vanishing amplitude, round beam, and large separation case. It is given by

$$\xi\frac{2}{d^6}\left\{2\left(d_x^3\eta_x + 3d_x^2 d_y\eta_y - 3d_x d_y^2\eta_x - d_y^3\eta_y\right)\right\}. \qquad (9)$$

3 APPLICATIONS TO THE TEVATRON

The tune shifts in the Tevatron are different from bunch to bunch [1]. For one $\bar{p}$ bunch ($\#\ 6$), using as separations the distances between the closed orbits at all the 72 encounters, we computed the amplitude dependent tune shifts using (3) and summed up over all collisions. Inspection of the separations reveals that over the 70 long-range interactions $6 \lessapprox \left|\vec{d}\right| \lessapprox 14$ which entail tune shifts of different signs. Computing the tune shifts at amplitudes of $(a_x, a_y) = (6, 6)$, we obtained the distribution depicted in Figure 1 (for the horizontal case; the vertical case is similar). The distributions look similar for intermediate amplitudes. The maximum long range tune shifts are encountered at the interactions which are closest to the interaction points. Using tune shift information from intermediate amplitude values, we got the tune footprint shown in Figure 2. Overall, there is a good agreement with tracking results [1]; the maximum difference is about 10^{-3}.

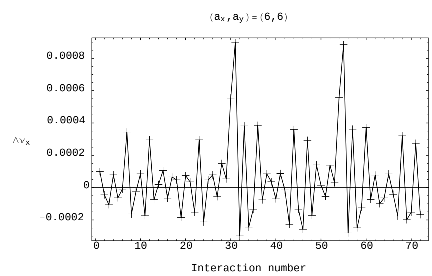

Figure 1: Horizontal amplitude dependent tune shifts of a particle with amplitudes $(a_x, a_y) = (6, 6)$. Head on collisions happen at interaction numbers 30 and 54, while maximum long range tune shifts are at interaction numbers 55 and 31.

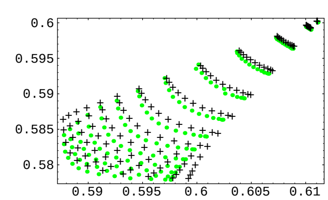

Figure 2: Tune footprints corresponding to all 72 interactions. Shown is a superposition of the analytical results with the tune footprints obtained by FFT of tracking data.

Among the long range collisions the nearest parasitic interactions dominate the tune shift contribution to the footprint. However, this is not true for the chromaticities. It can be shown [7] that the head on interactions do not generate linear chromatic effects, even if the dispersion does not vanish. The relative importance of the nearest parasitics and all long range can be seen from Figure 3. It is clear that the nearest parasitics are dominated by the rest of the parasitics.

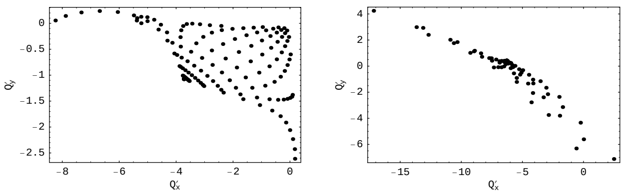

Figure 3: Comparison of two chromaticity footprints: (i) Nearest parasitics only. (ii) All beam-beam interactions included.

3.1 *Correction schemes*

Since tracking shows that the nearest parasitics dominate the nonlinear dynamics, we concentrated on these interactions only. We developed some compensation schemes for the tune and chromaticity footprints, and performed some studies aimed at unveiling correlations, if any, between this group of long range beam-beam interactions and the dynamic aperture.

The correction schemes are based on minimization of the footprints, by compensating for the aspect ratios or dispersions. The corresponding conditions were derived from the zero amplitude expressions of the appropriate relations, and hence it was not obvious a priori that the conditions are useful for non-zero amplitudes, the case which is too cumbersome to treat it analytically. For more details we refer the reader to [7]. However, compensation of aspect ratios clearly reduces both the shift and the spread of the tunes, as can be seen in Figure 4. On the other hand, compensation of aspect ratios does not have a dramatic effect on the chromaticity footprint. Perhaps more importantly, aspect ratio compensation does not harm the chromaticity footprint. The chromaticity footprint is mainly affected by compensation of the dispersions; there is a significant reduction in the size of the footprint. The results are contained in Figure 5.

Unfortunately, the tracking with the corrected footprints does not show the dramatic improvements in DA of the magnitude seen in the footprints. The DA increases by a maximum amount of around 0.6σ.

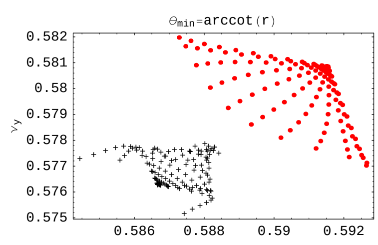

Figure 4: Tune footprint compensation of the nearest parasitic beam-beam interactions by aspect ratio compensation.

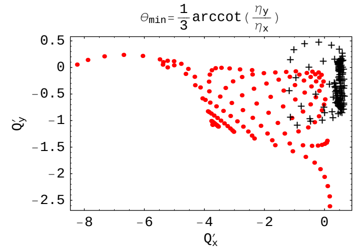

Figure 5: Chromaticity footprint compensation of the nearest parasitic beam-beam interactions by dispersion compensation.

4 CONCLUSIONS

We derived a useful analytical tool for the computation of the amplitude dependent tune shifts and linear chromaticities due to beam-beam interactions. The expressions can be used for efficient numerical evaluation at any amplitude, separation, dispersion, and aspect ratio. The favorable convergence properties make it especially suitable for studies of the parasitic beam-beam interactions.

We examined the impact of reducing the tune and chromaticity footprints of the nearest parasitics on the dynamic aperture. While the footprints can be significantly reduced by compensating for aspect ratios or dispersions, their effect on the dynamic aperture is less than satisfactory. However, the schemes do point to mechanisms which may increase the stable area available to the beam. For example, reducing the momentum spread in the beam and the linear chromaticity in the Tevatron at top energy would be helpful.

5 REFERENCES

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